On the m-Hull Number of the Join and Composition of Graphs

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ABSTRACT

In this paper, the $m$-hull sets in the join and composition of two connected graphs are characterized and their $m$-hull numbers are shown to be direct consequences of these characterizations.

1 Introduction

Given a connected graph $G = (V(G), E(G))$ and vertices $u$ and $v$ of $G$, we call any $u$-$v$ path of length $d_G(u, v)$ (length of the shortest path connecting $u$ and $v$) as $u$-$v$ geodesic. Any path that does not contain an edge joining two non-consecutive vertices $u$ and $v$ is called a $u$-$v$ monophonic path (or simply, $m$-path).

The monophonic closure of a subset $S$ of $V(G)$ is $J_G[S] = \bigcup_{u,v \in S} J_G[u,v]$, where $J_G[u,v]$ is the set containing $u$ and $v$ and all vertices lying on some $u$-$v$ $m$-path.

A subset $C$ of $V(G)$ is said to be $m$-convex if, for every pair of vertices $x, y \in C$, the vertex set of every $x$-$y$ $m$-path is contained in $C$. It is easy to verify that $S$ is $m$-convex if and only if $J_G[S] = S$.

The $m$-convex hull $[S]_m$ of a subset $S$ of $V(G)$ is the smallest $m$-convex set in $G$ containing $S$. It can be formed from the sequence $\{J_G^p[S]\}$, where $p$ is a nonnegative integer, $J_G^0[S] = S$, $J_G^1[S] = J_G[S]$, and $J_G^p[S] = J_G[J_G^{p-1}[S]]$ for $p \geq 2$. For some $p$, we must have $J_G^p[S] = J_G^q[S]$ for all $q \geq p$. Further, if $p$ is the smallest nonnegative integer such that $J_G^p[S] = J_G^q[S]$ for all $q \geq p$, then $J_G^p[S] = [S]_m$. If $[S]_m = V(G)$, then we call $S$ an $m$-hull set (monophonic hull set). The $m$-hull number of $G$, denoted by $mh(G)$, is the minimum cardinality of an $m$-hull set in $G$. Any $m$-hull set $S$ with $|S| = mh(G)$ is called a minimum $m$-hull set in $G$. 
The next remark that follows is from the definition of the $m$-convex hull of a set.

**Remark 1.1** Let $G$ be a connected graph and $S \subseteq V(G)$. Then $[S]_m = S$ if and only if $S$ is an $m$-convex set in $G$.

**Lemma 1.2** Let $G$ be a connected graph. Then $mh(G) = |V(G)|$ if and only if $G$ is a complete graph.

**Proof.** Let $G$ be a connected graph and suppose that $mh(G) = |V(G)|$. Suppose further that $G$ is not complete. Then there exist $a, b \in V(G)$ such that $d_G(a, b) \neq 1$. Let $c \in J_G[a, b] \setminus \{a, b\}$ and consider $S = V(G) \setminus \{c\}$. Then $S$ is not $m$-convex in $G$; hence $J_G[S] \neq S$. Thus, $S \subset J_G[S]$. It follows that $J_G[S] = V(G)$. This implies that $V(G) = J_G[S] \subseteq [S]_m$. Therefore, $[S]_m = V(G)$; that is, $S$ is an $m$-hull set in $G$. Hence, $mh(G) \leq |S| = |V(G)| - 1$, contrary to our assumption. Therefore, $G$ must be a complete graph.

For the converse, suppose that $G$ is a complete graph. Then every subset $C$ of $V(G)$ is $m$-convex in $G$; hence, $[C]_m = C$ by Remark 1.1. Thus, $[C]_m = V(G)$ if and only if $C = V(G)$. This implies that $C = V(G)$ is the only $m$-hull set in $G$. Thus, $mh(G) = |V(G)|$. ■

## 2 Join of Two Graphs

Our first goal is to characterize $m$-hull sets in $G + K_n$, where $G$ is a connected graph and $K_n$ the complete graph of order $n$.

**Lemma 2.1** Let $G$ be a connected non-complete graph, $S \subseteq V(G + K_n)$ and $S_1 = S \cap V(G)$. For every $p \geq 0$, if $y \in J_{G+K_n}^p[S \cap V(G)]$, then $y \in J_G^p[S_1]$.

**Proof.** The conclusion clearly holds for $p = 0$. Let $y \in J_{G+K_n}^p[S \cap V(G)]$. If $y \in S_1$, then $y \in J_G^p[S_1]$. Suppose $y \notin S_1$. Then there exist $s, t \in S$ such that $y \in J_{G+K_n}^p[s, t]$. Since $y \neq s$ and $y \neq t$, it follows that $d_{G+K_n}(s, t) = 2$. This means that $s, t \in S_1$. Since every $s-t$ $m$-path in $G + K_n$ containing $y$ is an $s-t$ $m$-path in $G$, it follows that $y \in J_G^p[s, t]$. Thus, $y \in J_G^p[S_1]$. This shows that the assertion of the Lemma holds for $p = 1$.

Next, suppose that the assertion holds for $p \geq 1$. Assume that $y \in J_{G+K_n}^{p+1}[S \cap V(G)]$. If $y \in J_{G+K_n}^p[S]$, then $y \in J_G^p[S_1]$ by assumption. So suppose that $y \notin J_{G+K_n}^p[S]$. Then there exist $u, v \in J_{G+K_n}^p[S]$ such that $y \in J_{G+K_n}^p[u, v]$. Since $y \neq u$ and $y \neq v$, it follows that $d_{G+K_n}(u, v) = 2$. Hence, $u, v \in V(G)$. By the assumption, $u, v \in J_G^p[S_1]$. This implies that $y \in J_G^{p+1}[S_1]$. Therefore, the assertion also holds for $p + 1$. This completes the proof of the lemma. ■

**Theorem 2.2** Let $G$ be a connected non-complete graph. A subset $S \subseteq V(G + K_n)$ is an $m$-hull set in $G + K_n$ if and only if $S \cap V(G)$ is an $m$-hull set in $G$. 2
Proof. Let $G$ be a connected non-complete graph. Suppose $S \subseteq V(G + K_n)$ is an $m$-hull set in $G + K_n$, say $[S]_m = J_{G + K_n}^p[S] = V(G + K_n)$, where $p$ is the smallest non-negative integer satisfying the first equality. Let $S_1 = S \cap V(G)$ and let $x \in V(G) \setminus J_{G}^{p-1}[S]_1$. Since $S$ is an $m$-hull set in $G + K_n$, there exist $a, b \in J_{G + K_n}^p[S]$ such that $x \in J_{G + K_n}[a, b]$. By the contrapositive of Lemma 2.1, $x \notin J_{G + K_n}^p[S]$; hence, $x \neq a$ and $x \neq b$. This implies that $d_{G + K_n}(a, b) = 2$, that is, $a, b \in V(G)$. Thus, $a, b \notin J_{G}^{p-1}[S]_1$ by Lemma 2.1. Therefore, $x \notin J_{G}^{p-1}[S]_1$. This shows that $V(G) \setminus J_{G}^{p-1}[S]_1 \subseteq J_{G}^p[S]_1$. Since $J_{G}^{p-1}[S]_1 \subseteq J_{G}^p[S]_1$, it follows that $V(G) \subseteq J_{G}^p[S]_1$. Therefore, $S_1$ is an $m$-hull set in $G + K_n$.

For the converse, suppose that $S_1$ is an $m$-hull set in $G$, say $J_{G}^p[S_1] = V(G)$, where $p$ is the smallest nonnegative integer satisfying the equality. Since $J_{G}^p[S_1] \subseteq J_{G + K_n}^p[S_1]$, it follows that $V(G) \subseteq J_{G + K_n}^p[S_1]$. If $(S_1)$ were a complete subgraph of $G$, then $J_{G}^p[S_1] = S_1$ for all nonnegative integer $r$. Thus, $J_{G}^p[S_1] = S_1 \neq V(G)$ contrary to our assumption. Therefore, $(S_1)$ is non-complete. Choose $u, v \in S_1$ such that $d_G(u, v) \neq 1$. Then $d_{G + K_n}(u, v) = 2$ and $[u, w, v]$ is a $u-v$ $m$-path for every $w \in V(K_n)$. It follows that $V(K_n) \subseteq J_{G + K_n}[u, v] \subseteq J_{G + K_n}^p[S_1]$. Therefore, $V(G + K_n) \subseteq J_{G + K_n}^p[S_1]$, that is, $S_1$ is an $m$-hull set in $G + K_n$. Since $S_1 \subseteq S$, $S$ is also a $m$-hull set in $G$. This completes the proof of the theorem.

Corollary 2.3 Let $G$ be a connected non-complete graph. A subset $S \subseteq V(G + K_n)$ is a minimum $m$-hull set in $G + K_n$ if and only if $S \subseteq V(G)$ and $S$ is a minimum $m$-hull set.

Proof. Let $G$ be a connected non-complete graph. Suppose $S \subseteq V(G + K_n)$ is a minimum $m$-hull set in $G + K_n$. Then $S_1 = S \cap V(G)$ is an $m$-hull set in $G$ by Theorem 2.2. Suppose that $S^* = S \cap V(K_n) \neq \emptyset$. Then $S_1 = S \setminus S^*$ is an $m$-hull set in $G + K_n$ by Theorem 2.2 and $|S_1| < |S|$. This contradicts the fact that $S$ is a minimum $m$-hull set in $G + K_n$. Therefore, $S \subseteq V(G)$. Again, by Theorem 2.2, $S$ is a minimum $m$-hull set in $G$, that is, $mh(G) = |S|$.

The converse can be proved in a similar manner.

The following result gives the $m$-hull number of $G + K_n$ which is a direct consequence of the above Corollary and Lemma 1.2

Corollary 2.4 Let $G$ be a connected graph of order $p$ and $K_n$ the complete graph of order $n$. Then

$$mh(G + K_n) = \begin{cases} p + n & \text{if } G = K_p, \\ mh(G) & \text{if } G \neq K_p. \end{cases}$$
Example 2.5 Let \( n \) and \( m \) be positive integers. Then

1. \( mh(K_n + K_m) = n + m \) for all \( n \geq 2 \);
2. \( mh(C_n + K_m) = 2 \) for all \( n \geq 4 \);
3. \( mh(W_n + K_m) = 2 \) for all \( n \geq 4 \).

The next result characterizes \( m \)-hull sets in the join of any non-complete graphs.

**Theorem 2.6** Let \( G \) and \( H \) be non-complete graphs. A subset \( S \) of \( V(G + H) \) is an \( m \)-hull set in \( G + H \) if and only if there exist \( a, b \in S \) with \( d_{G+H}(a, b) = 2 \). In this case, either \( a, b \in V(G) \) or \( a, b \in V(H) \).

**Proof.** Let \( G \) and \( H \) be non-complete graphs and \( S \subseteq V(G + H) \) an \( m \)-hull set in \( G + H \). If \( S = V(G + H) \), then we are done. So suppose \( S \neq V(G + H) \). Since \( S \) is an \( m \)-hull set in \( G + H \), it follows that \( S \neq J_{G+H}[S] \) (otherwise, \( J_{G+H}^p[S] = S \neq V(G + H) \) for all \( p \geq 1 \), contrary to our assumption). Let \( x \in J_{G+H}[S] \setminus S \). Then there exist \( a, b \in S \) such that \( x \in J_{G+H}[a, b] \). Since \( x \neq a \) and \( x \neq b \), \( d_{G+H}(a, b) = 2 \). In this case, \( a, b \in V(G) \) or \( a, b \in V(H) \).

Conversely, suppose that there exist \( a, b \in S \) with \( d_{G+H}(a, b) = 2 \). Assume that \( a, b \in V(G) \). Then \( V(H) \subseteq J_{G+H}[a, b] \subseteq J_{G+H}[S] \). Pick \( u, v \in V(H) = J_{G+H}[S] \) such that \( uv \notin E(H) \). Then \( d_{G+H}(u, v) = 2 \); hence \( V(G) \subseteq J_{G+H}^2[S] \). Since \( J_{G+H}[S] \subseteq J_{G+H}^2[S] \), \( V(G + H) \subseteq J_{G+H}^2[S] \). This shows that \( S \) is a \( m \)-hull set in \( G + H \).

As a quick consequence of Theorem 2.6, we have

**Corollary 2.7** Let \( G \) and \( H \) be non-complete graphs. Then \( mh(G + H) = 2 \).

### 3 Compositon of Two Graphs

The concept that follows appeared and was illustrated in [5].

**Definition 3.1** Let \( G \) be a connected graph and \( A \subseteq V(G) \). A point \( a \in A \) is called a monophonic interior point of \( A \) if \( a \in J_G[A \setminus \{a\}] \). The set of all monophonic interior points of \( A \) is denoted by \( A^o \).

**Lemma 3.2** [5]. Let \( G \) be a connected graph and \( K_n \) be the complete graph of order \( n \). If \( P = [(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)] \), \( r \geq 2 \) is an \( m \)-path in \( G[K_n] \), then we have the following possibilities:

i. If \( u_i's \) are distinct, then \( [u_1, u_2, \ldots, u_r] \) is an \( m \)-path in \( G \).

ii. If \( u_i's \) are not distinct, then \( r = 2 \).
Lemma 3.3  Let $G$ be a connected graph and $K_n$ the complete graph of order $n$, and $C \subseteq V(G[K_n])$. Then $(J_{G[K_n]}^k[C])_G = J_G^k[C_G]$.

Proof. We prove this by induction on $k$.

Let $a \in (J_{G[K_n]}[C])_G$. Then there exists $b \in V(K_n)$ such that $(a, b) \in (J_{G[K_n]}[C])_G$. Thus, we can find an $m$-path $[(u_1, v_1), \ldots, (u_s, v_s)]$ in $G[K_n]$ such that $(u_1, v_1), (u_s, v_s) \in C$ and $(a, b) = (u_r, v_r)$ for some $r, 1 \leq r \leq s$. If $s = 2$, then clearly, $[u_1, u_2]$ is an $m$-path in $G$. Thus $a \in C_G$. Suppose $s \geq 3$, then by Lemma 3.2, $[u_1, \ldots, u_s]$ is an $m$-path in $G$ containing $a$. Thus, in either case, $a \in J_G[C_G]$.

Conversely, let $a \in J_G[C_G]$. Then there exists an $m$-path $[u_1, \ldots, u_s] \in G$ such that $u_1, u_s \in C$ and $a = u_r$, for some $r, 1 \leq r \leq s$. Let $v, v' \in V(K_n)$ such that $(u_1, v), (u_s, v') \in C$. If $v = v'$, then $[(u_1, v), \ldots, (u_s, v')$ is an $m$-path in $G[K_n]$ containing $(a, v)$. If $v \neq v'$, then $[(u_1, v), \ldots, (u_s, v'), (u_s, v')]$ is an $m$-path in $G[K_n]$ containing $(a, v)$. Thus, $(a, v) \in (J_{G[K_n]}[C])_G$. Consequently, $a \in (J_{G[K_n]}[C])_G$.

This shows that the assertion holds for $k = 1$. Next, suppose the assertion holds for $k = n > 1$. Now $(J_{G[K_n]}^{n+1}[C])_G = J_G^{n+1}[C_G]$. By induction hypothesis, $J(J_{G[K_n]}^{n+1}[C])_G = J_G^{n+1}[C_G]$. Thus $(J_{G[K_n]}^{n+1}[C])_G = J_G^{n+1}[C_G]$. Therefore, the assertion holds for all integer $k \geq 1$.

Theorem 3.4  [5]. Let $G$ be a connected graph and $K_n$ the complete graph of order $n$. Then $C = \bigcup_{a \in S} \{(a) \times T_a\}$, where $S \subseteq V(G)$ and $T_a \subseteq V(K_n)$, is monophonic in $G[K_n]$ if and only if $S$ is monophonic in $G$ and $T_a = V(K_n)$ whenever $a \in S \setminus S^o$.

The next result describes completely the $m$-hull sets in $G[K_n]$.

Theorem 3.5  Let $G$ be a connected graph, $K_n$ the complete graph of order $n$, and $C \subseteq V(G[K_n])$. Then $C = \bigcup_{a \in S} \{(a) \times T_a\}$ is an $m$-hull set in $G[K_n]$ if and only if $S$ is an $m$-hull set in $G$ if $T_a = V(K_n)$ whenever $a \in S \setminus (V(G))^{o}$, where $(V(G))^o$ is the set of all monophonic interior points of $G$.


Thus, $J_G^0[S] = J_G^{l-1}[S] = V(G)$, i.e., $S$ is an $m$-hull set in $G$.

Next, $J_G^{l-1}[C]$ monophonic in $G[K_n]$ implies $(J_{G[K_n]}^{l-1}[C])_G$ is monophonic in $G$ and $T_a = V(K_n)$ whenever $a \in (J_{G[K_n]}^{l-1}[C])_G \setminus (J_{G[K_n]}^{l-1}[C])_G^o$ by Theorem 3.4. This implies that $J_G^{l-1}[S]$ is monophonic in $G$ and $T_a = V(K_n)$ whenever $a \in J_G^{l-1}[S] \setminus (J_G^{l-1}[S])^o$. Consequently, $J_G^l[S] = V(G)$ and $T_a = V(K_n)$.
whenever $a \in J^{l-1}_G[S] \setminus (J^l_G[S])^\circ$. Note that $S \subseteq J^{l-1}_G[S]$ and $(J^l_G[S])^\circ \subseteq V(G)^\circ$. It follows that $J^l_G[S] = V(G)$ and $T_a = V(K_n)$ whenever $a \in S \setminus V(G)^\circ$.

For the converse, suppose $S$ is an $m$-hull set in $G$ and $T_a = V(K_n)$ whenever $a \in S \setminus V(G)^\circ$. Then $V(G) = J^l_G[S] = (J^l_G[K_n]|C)_G$. Let $(a, b) \in V(G[K_n])$. If $a \in S \setminus V(G)^\circ$, then $(a, b) \in C \subseteq J^l_G[K_n]|C]$. Suppose $a \in (V(G))^\circ$. Then there exist $u, u' \in V(G) \setminus \{a\}$ such that $a \in J_G[u, u']$. Let $[u_1, \ldots, u_r]$ be an $m$-path in $G$ such that $u_1 = u, u_r = a$, and $u_s = u'$. Let $v, v' \in V(K_n)$ such that $(u, v), (u'v') \in J^l_G[K_n]|C]$. Consider the following cases:

Case 1 $v = v' = b$. Then $[(u_1, v), \ldots, (u_r, v), (u_s, v)]$ is an $m$-path in $G[K_n]$ containing $(a, b)$.

Case 2 $v \neq b$ and $v' = b$. Then $[(u_1, v), \ldots, (u_r, v), (u_r, b), \ldots, (u_s, b)]$ is an $m$-path in $G[K_n]$ containing $(a, b)$.

Case 3 $v = b$ and $v' \neq b$. This case is similar to Case 2.

Case 4 $v \neq b$ and $v' \neq b$. Then $[(u_1, v), \ldots, (u_r, v), (u_r, b), \ldots, (u_s, b), (u_s, v')]$ is an $m$-path in $G[K_n]$ containing $(a, b)$.

Thus $(a, b) \in J^l_G[K_n]|C$.

Therefore $V(G[K_n]) = J^{l+1}_G[K_n]|C]$. Accordingly, $C$ is an $m$-hull set in $G[K_n]$.

The next result gives formula for computing the $m$-hull number of $G[K_n]$.

**Corollary 3.6** Let $G$ be a connected graph, $K_n$ the complete graph of order $n$, and $(V(G))^\circ$ be the set of all monophonic interior points of $V(G)$. Then

$$mh(G[K_n]) = \min\{n|S \setminus (V(G))^\circ| + |S \cap (V(G))^\circ| : S \text{ is an } m\text{-hull set in } G\}.$$

We now give the $m$-hull number of the composition $G[H]$, where $G$ and $H$ are non-complete graphs.

**Theorem 3.7** Let $G$ and $H$ be non-complete graphs. Then $mh(G[H]) = 2$.

**Proof.** Let $a \in V(G)$ and $b, b' \in V(H)$ such that $d_H(b, b') = 2$. We show that $S = \{(a, b), (a, b')\}$ is an $m$-hull set in $G[H]$. Let $(u, v) \in V(G[H])$ such that $u \neq a$ and $[u_1, u_2, \ldots, u_r]$ be an $a$-$u$ geodesic, where $u_1 = a$ and $u_r = u$.

**Claim:** For each $(u, v) \in V(G[H])$, there exists $l(u)$ such that $(u, v) \in J^l[S]$.

Since $(u_1, b), (u_1, b') \in S$, $(u_2, b), (u_2, b') \in J^2[S]$. Continuing in this manner, we obtain $(u_{r-1}, b), (u_{r-1}, b') \in J^{r-2}[S], r \geq 2$. Consequently, $(u_r, v) = (u, v) \in J^{r-1}[S]$.

Let $n = \max\{l(u) : u \in V(G)\}$ and $(u, v) \in V(G[H])$. Then by the Claim, $(u, v) \in J^{n-1}[S] \subseteq J^n[S]$. Thus, $V(G[H]) = J^n[S]$. Accordingly, $S$ is an $m$-hull set in $G[H]$. Thus, $mh(G[H]) = 2$. 

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6
References


